

Physics in one dimension: Theoretical concepts for quantum many body systems

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Abstract.

Various sophisticated approximation methods exist for the description of quantum many body systems. It was realized early that the theoretical description can simplify considerably in one dimensional systems and various exact solutions exist. The focus in this introductory paper is on fermionic systems and the emergence of the Luttinger liquid concept.

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1. Introduction

The theoretical description of systems of interacting particles is notoriously difficult and only few exact results are available. Especially in the development of the theory of continuous phase transitions it became obvious that the number d of spatial dimensions plays an important role. While for most experimentally realizable systems one has $d = 3$, it early turned out that for $d = 1$, i.e., one-dimensional systems the theoretical description simplifies considerably and exact results for various types of spin chains can be obtained. By a mapping to fermions this implies exact results also for special lattice models of interacting “electrons” in one dimension as discussed in section 2.

In this introductory paper the focus is on (normal) one-dimensional fermionic *quantum liquids*, which are many body systems in which the indistinguishability of the elementary constituents is important. These particles can live on a lattice or the continuous line. On low energy scales the metallic state is a non-Fermi liquid characterized by a power law decay of space-time correlation functions with interaction dependent exponents. The name *Luttinger liquid* (LL) was termed for this behaviour [1].

While in the beginning of the theoretical developments the corresponding models were considered a mere playground for theoreticians recent developments have made the attempt to experimentally verify Luttinger liquid behaviour a flourishing field of research as shown in this special issue.

This paper gives a historical account of the emergence of the Luttinger liquid concept. The typical properties were first found in the models proposed by Tomonaga [2] and Luttinger [3], which use rather restrictive assumptions about the interaction. Tomonaga’s important step was to realize and use the fact that the low energy spectrum of noninteracting fermions in one dimension is identical to that of a harmonic chain. This allows to describe the interacting fermions as a system of coupled oscillators [2]. Luttinger’s calculation of the momentum distribution in the groundstate marks the appearance of power laws for interacting fermions in one dimension [3]. It was realized much later that the low energy physics of these models is generic under rather weak assumptions [1, 4]. This low energy physics can be found also in bosonic many body systems [4, 5]. Here the focus is on the analytical description of fermionic systems. Important computational techniques for one-dimensional quantum many body systems like the density matrix renormalization group (DMRG) [6, 7] are not discussed here.

2. Models

In this section we present models which played an important role for the theoretical understanding of interacting quantum systems in one dimension. We begin with the anisotropic spin 1/2 chain with nearest neighbour interaction in an external field h . The Hamiltonian reads

$$H = \sum_i \left[J_x s_i^x s_{i+1}^x + J_y s_i^y s_{i+1}^y + J_z s_i^z s_{i+1}^z - h s_i^z \right] , \quad (1)$$

with the exchange couplings J_α . The operators of the spin components on the same site i obey the usual angular momentum commutation relations and spin operators on different sites commute. For $J_x = J_y = 0$ one obtains the *Ising chain* [8], the simplest model of interacting spins. As all operators commute it can be considered a classical spin model. Ising's exact solution for the free energy showed that at finite temperatures no symmetry breaking to a state with a finite magnetization occurs for vanishing external field. It was suspected that this a special property of one dimension [9], later confirmed by Onsager's exact solution for the free energy of the *two-dimensional* Ising model with $h = 0$, showing a phase transition at finite temperature [10].

For the isotropic case $J_x = J_y = J_z = J$ and putting $h = 0$ one obtains the Hamiltonian of the Heisenberg spin 1/2 chain [11]. For $J < 0$ the ground state is ferromagnetically ordered, while for $J > 0$ the spins are antiferromagnetically correlated but not ordered in the groundstate. In 1931 Bethe [12] presented his famous Ansatz for the exact eigenstates and eigenvalues of this model which was later generalized to a larger class of 1d-models as discussed in various textbooks [13, 4, 14]. At the end of his paper Bethe announced a follow up paper with a generalization of his Ansatz to higher dimensions. The fact that it never appeared, again shows that many body physics in one dimension is special. In the following the sophisticated Bethe-Ansatz technique is not described, only exact results for Luttinger liquid parameters for lattice models are mentioned later. Unfortunately no exact results for the correlation functions discussed later are available within the Bethe Ansatz approach, but it should be mentioned that significant progress has been made recently to calculate dynamical spin structure factors using a non-perturbative ("form factor") approach [15, 16, 17].

Another special case of the general spin model in Eq. (1) should be mentioned. For $J_y = J_z = 0$ one obtains the *transverse Ising model* [18] which now serves as a standard model for a system with a quantum phase transition [19]. It was solved exactly [18] by the use of a Jordan-Wigner transformation [20, 21] which relates the set of spin 1/2 operators to a set of spinless Fermi operators. For $J_x = J_y$ the spin model in Eq. (1) reads in the fermionic representation

$$H = \sum_i \left[\epsilon_0 (c_i^\dagger c_i - \frac{1}{2}) - t (c_i^\dagger c_{i+1} + H.c.) + U (c_i^\dagger c_i - \frac{1}{2}) (c_{i+1}^\dagger c_{i+1} - \frac{1}{2}) \right], \quad (2)$$

where $\epsilon_0 = -h$, $t = -J_x/2$, $U = J_z$ and $c_i^{(\dagger)}$ is the annihilation (creation) operator of a fermion at site i . This Hamiltonian describes spinless fermions on a chain with a nearest neighbour "Coulomb interaction". The spin model with $J_z = 0$ ("XY-model") corresponds to noninteracting fermions and can therefore be solved exactly [21].

The spinless model in Eq. (2) is one of the lattice models which played an important role in the emergence of the Luttinger liquid concept. The other one is the one-dimensional Hubbard model [22, 23, 14]

$$H = \sum_{i,\sigma} \left[-t (c_{i,\sigma}^\dagger c_{i+1,\sigma} + H.c.) + U c_{i,\uparrow}^\dagger c_{i,\uparrow} c_{i,\downarrow}^\dagger c_{i,\downarrow} \right], \quad (3)$$

where σ is the spin label and U the on-site Coulomb repulsion.

In his seminal paper Tomonaga [2] treated interacting fermions on the continuous line. Without impurities all fermionic models discussed in this paper can be written in the form

$$H = \sum_k \epsilon_k c_k^\dagger c_k + \frac{1}{2} \sum_{k_1, k_2, k_3, k_4} v_{k_1 k_2 k_3 k_4} c_{k_1}^\dagger c_{k_2}^\dagger c_{k_4} c_{k_3}, \quad (4)$$

where k is a double index k, σ for models including spin like the Hubbard model. For the lattice models Eq. (2) and Eq. (3) the momenta k_i are in the first Brillouin zone and for continuum models they are on the line extending from $-\infty$ to ∞ . The energy dispersion ϵ_k and the interaction matrix elements $v_{k_1 k_2 k_3 k_4}$ are specified in the following sections.

3. The Tomonaga-Luttinger model

A decisive step towards an understanding of interacting fermions in one dimension beyond perturbation theory was Tomonaga's idea [2] to *bosonize* the Hamiltonian Eq. (4) for nonrelativistic particles on a line L with periodic boundary conditions ($\hbar = 1$)

$$\epsilon_k = k^2/(2m), \quad v_{k_1 k_2 k_3 k_4} = \frac{1}{L} \tilde{v}(k_1 - k_3) \delta_{k_1 + k_2, k_3 + k_4}. \quad (5)$$

Tomonaga studied the case when the Fourier transform of the two-body interaction $\tilde{v}(k)$ is nonzero only for $|k| < k_c \equiv 2\pi n_c/L$, where the cut-off k_c is much smaller than the the Fermi momentum $k_F \equiv 2\pi n_F/L$. This corresponds to a long range interaction in real space. Perturbation theory then indicates that the ground state and *low energy* excited states have negligible admixtures of holes deep in the Fermi sea and particles with momenta $|k| - k_F \gg k_c$. This is the motivation for Tomonaga's approximation to *linearize* the dispersion ϵ_k in the regions around the two Fermi points $\pm k_F$, with particle-hole pairs present

$$k \approx \pm k_F : \quad \epsilon_k = \epsilon_F \pm v_F(k \mp k_F), \quad (6)$$

with $v_F = k_F/m$ the Fermi velocity. Tomonaga realized that the Fourier components of the operator of the density

$$\hat{\rho}_n = \int_{-L/2}^{L/2} \hat{\rho}(x) e^{-ik_n x} dx = \sum_{n'} c_{n'}^\dagger c_{n'+n}, \quad (7)$$

where $c_{n'}^\dagger (c_{n'})$ creates (annihilates) a fermion in the state with momentum $k_{n'} = 2\pi n'/L$, play a central role not only for the interaction but also for the *kinetic energy*. His important idea was to split $\hat{\rho}_n$ for momentum transfer $|k_n| \ll k_F$ into two parts, one containing operators of “*right movers*” i.e. involving fermions near the right Fermi point k_F with velocity v_F and “*left movers*” involving fermions near $-k_F$ with velocity $-v_F$

$$\hat{\rho}_n = \sum_{n' > 0} c_{n'}^\dagger c_{n'+n} + \sum_{n' \leq 0} c_{n'}^\dagger c_{n'+n} \equiv \hat{\rho}_{n,+} + \hat{\rho}_{n,-}. \quad (8)$$

In the subspace with *no holes deep in the Fermi sea* in which all one-particle states $|k_j\rangle$ with $|j| \leq M = n_F - \gamma n_c$ are occupied, the commutation relations [2]

$$[\hat{\rho}_{m,\alpha}, \hat{\rho}_{n,\beta}] = \alpha m \delta_{\alpha\beta} \delta_{m,-n} \hat{1} \quad (9)$$

hold for $|n|, |m| \leq M$. The dimensionless constant γ has to be chosen larger for stronger interaction. If one defines the operators

$$b_n \equiv \frac{1}{\sqrt{|n|}} \begin{cases} \hat{\rho}_{n,+} & \text{for } n > 0 \\ \hat{\rho}_{n,-} & \text{for } n < 0 \end{cases} \quad (10)$$

and the corresponding adjoint operators b_n^\dagger this leads using $\rho_{n,\alpha}^\dagger = \rho_{-n,\alpha}$ to the bosonic commutation relations

$$[b_n, b_m] = 0, \quad [b_n, b_m^\dagger] = \delta_{mn} \hat{1}. \quad (11)$$

The kinetic energy of the right movers as well as that of the left movers can be expressed as a bilinear form of the $b^{(\dagger)}$ -operators using a remarkable operator identity first presented by Kronig in a different context [24, 25, 26]. For the right movers it reads

$$T_+ = \sum_{n=1}^{\infty} v_F k_n c_n^\dagger c_n = v_F \frac{2\pi}{L} \left[\sum_{m=1}^{\infty} m b_m^\dagger b_m + \frac{1}{2} \mathcal{N}_+ (\mathcal{N}_+ + 1) \right], \quad (12)$$

where $\mathcal{N}_+ = \sum_{n=1}^{\infty} c_n^\dagger c_n$ is the particle number of the right movers. For its proof the commutation relations Eq. (11) have *not* to be used.

As \hat{V} is bilinear in the $\hat{\rho}_n$ the same is true for the $\hat{\rho}_{n,\alpha}$. For the linearized fermionic dispersion $\epsilon_k = v_F |k| + \text{const.}$ shown as the dashed line in Fig. 1 and the two-body interaction in Eq. (5) the Hamiltonian in Eq. (4) can therefore apart from an additional term *linear* in the particle number operators \mathcal{N}_\pm exactly be rewritten as

$$\begin{aligned} \tilde{H} = \sum_{n>0} k_n \left\{ \left(v_F + \frac{\tilde{v}(k_n)}{2\pi} \right) (b_n^\dagger b_n + b_{-n}^\dagger b_{-n}) \right. \\ \left. + \frac{\tilde{v}(k_n)}{2\pi} (b_n^\dagger b_{-n}^\dagger + b_{-n} b_n) \right\} + \frac{\pi}{2L} [v_N \mathcal{N}^2 + v_J \mathcal{J}^2] \equiv H_B + H_{\mathcal{N}, \mathcal{J}}, \end{aligned} \quad (13)$$

where $\mathcal{N} \equiv \mathcal{N}_+ + \mathcal{N}_-$ is the total particle number operator, $\mathcal{J} \equiv \mathcal{N}_+ - \mathcal{N}_-$ the “current operator”, and the velocities are given by $v_N = v_F + \tilde{v}(0)/\pi$ and $v_J = v_F$.

If one now *assumes* that the bosonic commutation relations in Eq. (11) hold generally (see discussion below) the operators H_B and $H_{\mathcal{N}, \mathcal{J}}$ commute and with the Bogoliubov transformation $\alpha_n^\dagger = b_n^\dagger \cosh \theta_n - b_{-n} \sinh \theta_n$ to new boson operators the Hamiltonian H_B can be brought into the form

$$H_B = \sum_{n \neq 0} \omega(k_n) \alpha_n^\dagger \alpha_n + \text{const.}, \quad \omega(k_n) = v_F |k_n| \sqrt{1 + \tilde{v}(k_n)/(\pi v_F)} \quad (14)$$

and θ_n is determined by

$$\tanh \theta_n = -\tilde{v}(k_n)/(2\pi v_F + \tilde{v}(k_n)). \quad (15)$$

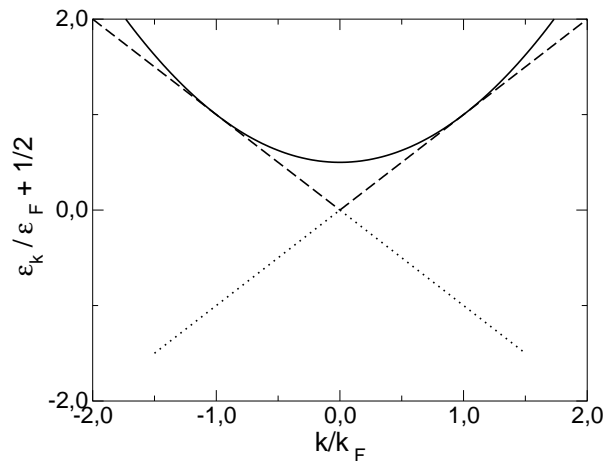


Figure 1. Energy dispersion as a function of momentum. The full curve shows the usual “nonrelativistic” dispersion and the dashed curve the linearized version. The dot-dashed parts are the additional states for $k_{\text{Band}} = 1.5k_F$. The model discussed by Luttinger corresponds to $k_{\text{Band}} \rightarrow \infty$.

For $|k_n| \ll k_c$ and a smooth $\tilde{v}(k)$ the *boson dispersion* is approximately linear $\omega(k_n) \approx v_c|k_n|$ with $v_c = \sqrt{v_N v_J}$ the *charge velocity*. With the approximation to linearize ϵ_k around the Fermi points it is strictly linear up to k_c if $\tilde{v}(k)$ is constant up to the cut-off. Besides the charge velocity v_c the “stiffness constant” $K \equiv \sqrt{v_J/v_N}$ plays an important role. The noninteracting case yields $K = 1$, attractive interactions $\tilde{v}(0) < 0$ lead to $K > 1$ and repulsive interaction with $\tilde{v}(0) > 0$ imply $0 < K < 1$. For the generalized model, where $\tilde{v}(k_n)$ is replaced by $g_4(k_n)$ in the first line on the rhs of Eq. (13) (scattering events on one of the Fermi points) and by $g_2(k_n)$ in the second line (scattering events involving both Fermi points which conserve the number of right and left movers) the two *independent* quantities v_c and K describe the low energy physics. This generalization to non-Galilei-invariant systems turns out to be important for the general Luttinger liquid concept discussed in section 4.

A simple trick to extend the range of validity of Eq. (11) and therefore for the step from Eq. (13) to Eq. (14) is to make the Fermi sea *deeper* for fixed k_c by extending the left (right) mover branch to positive (negative) k -values up to $k_{\text{Band}} > 0$ ($-k_{\text{Band}} < 0$) as shown in Fig. 1 for the (arbitrary) value $k_{\text{Band}} = 1.5k_F$. The Kronig relation is easily extended to this case and leads to an additional term linear in the particle number operators and therefore Eq. (13) still holds. Luttinger treated a model with two *infinite* branches of right and left moving fermions with dispersion $\pm v_F k$ [3]. As he made an error related to the fact that his Hamiltonian is not bounded from below, it is better to switch from Tomonaga’s to Luttinger’s model keeping k_{Band} finite before taking the limit $k_{\text{Band}} \rightarrow \infty$ [27]. Because of the close relation of both models the term “*Tomonaga-Luttinger (TL) model*” is often used.

Fortunately Luttinger’s error had no influence on his inquiry if a discontinuity of

$\langle n_{k,+} \rangle$ at k_F exists in the exact ground state of the interacting model, as expected from Fermi liquid theory [28, 29]. After a lengthy calculation using properties of “Toeplitz determinants” Luttinger found that the average occupation $\langle n_{k,+} \rangle$ in the ground state for $k \approx k_F$ in the limit $L \rightarrow \infty$ behaves as

$$\langle n_{k,+} \rangle - \frac{1}{2} \sim \left| \frac{k - k_F}{k_c} \right|^\alpha \text{sign}(k_F - k), \quad (16)$$

where $\alpha \geq 0$ depends on the interaction strength (see below). This is the power law behaviour mentioned in the introduction. It cannot be obtained by finite order perturbation theory in the interaction strength, which produces logarithmic terms in $|k - k_F|$.

Luttinger’s error was corrected by Mattis and Lieb [30] who presented a new algebraic method to calculate $\langle n_{k,+} \rangle$. They showed that Eq. (16) only holds for $\alpha < 1$. For $\alpha > 1$ a term linear in $k - k_F$ dominates $\langle n_{k,+} \rangle - 1/2$. They also pointed out that one obtains $\langle n_{k,+} \rangle \equiv 1/2$ in the limit interaction cutoff $k_c \rightarrow \infty$ for a k -independent interaction. In this limit Luttinger’s model is equivalent to the massless Thirring model [31] and can be written as a (quadratic) local bosonic field theory [1, 4], not discussed here further.

Additional information about the system is encoded in its time dependent correlation functions $\langle A(t)B \rangle$, where $\langle \dots \rangle$ denotes the expectation value in the ground state (or in thermal equilibrium) and $A(t) = e^{iHt} A e^{-iHt}$ is the operator in the Heisenberg picture. As Eq. (14) implies $\alpha_n(t) = e^{-i\omega(k_n)t} \alpha_n$ the correlation function $\langle \hat{\rho}_n(t) \hat{\rho}_{-n} \rangle$ can easily be calculated using the inverse Bogoliubov transformation. Apart from a prefactor its Fourier transform in time is the *dynamical structure factor* $S(q, \omega)$ [32] at $q = k_n$. For the Hamiltonian Eq. (14) $S(q, \omega)$ is proportional to $|q| \delta(\omega - \omega(q))$. For the nonrelativistic dispersion $\epsilon_k = k^2/(2m)$ this can only be asymptotically correct for $q \rightarrow 0$. This can already be seen from the exact calculation of $S(q, \omega)$ for the noninteracting case. As $\epsilon_{k_F+q} - \epsilon_{k_F} = qv_F + q^2/(2m)$ and $\epsilon_{k_F} - \epsilon_{k_F-q} = qv_F - q^2/(2m)$, for $0 < q \ll k_F$ the dynamical structure factor for fixed q as function of frequency takes the form of a narrow box centered at $v_F q$ of width q^2/m and height $\sim 1/q$.

In the calculation of one-particle Green functions the Heisenberg operator $c_n(t)$ enters. It cannot be expressed using the “first step” of bosonization introduced by Tomonaga [2]. Even time independent expectation values like the momentum distribution $\langle n_{k,+} \rangle = \langle c_{k,+}^\dagger c_{k,+} \rangle$ require an additional theoretical concept as used by Luttinger [3] and Mattis and Lieb [30].

The calculation of $\langle n_{k,+} \rangle$ can be further simplified by *bosonizing the field operator*. This concept was introduced by Schotte and Schotte in the context of x-ray absorption from a core hole in the presence of a Fermi sea [33]. To the calculation of correlation functions of the Tomonaga-Luttinger-model it was first applied by Luther and Peschel [34]. Later subtleties of this second step of bosonization were addressed [1, 35, 26].

In this step the $c_{n\pm}$ are not bosonized directly but the field operators $\psi_\pm(x)$, e.g. for

the right movers

$$\psi_+(x) \equiv \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} e^{ik_n x} c_{n,+} , \quad (17)$$

where the limit $k_{\text{Band}} \rightarrow \infty$ apparently is performed first. The commutation relations of $\psi_+(x)$ with the boson operators b_m and b_m^\dagger imply the form

$$\psi_+(x) = \hat{O}_+(x) e^{i\phi_+^\dagger(x)} e^{i\phi_+(x)}, \quad i\phi_+(x) = \sum_{n=1}^{\infty} \frac{e^{ik_n x}}{\sqrt{n}} b_n , \quad (18)$$

where the *Klein operator* $\hat{O}_+(x)$ lowers the fermion number by one and commutes with all boson operators. Its explicit form has been discussed in detail [1, 35, 26]. It is not presented here, only the fact that $\hat{O}_+(x)$ and $\hat{O}_-(x')$ anticommute is mentioned.

Using Eq. (18), the Bogoliubov transformation and the Baker-Hausdorff formula, $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ if the operators A and B commute with $[A, B]$, it is straightforward to calculate ground state one-particle Green functions like $iG_+^<(x, t) \equiv \langle \psi_+^\dagger(0, 0) \psi_+(x, t) \rangle$ which enter the description of photoemission. Using $\alpha_n |E_0\rangle = 0$ one obtains [34]

$$ie^{i\mu t} G_+^<(x, t) = \frac{e^{ik_F x}}{L} \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \left[e^{-i(k_n x - \omega_n t)} + 2 \sinh^2 \theta_n (\cos(k_n x) e^{i\omega_n t} - 1) \right] \right\} , \quad (19)$$

where μ is the chemical potential. Putting $t = 0$ the momentum distribution $\langle n_{k,+} \rangle$ is obtained by Fourier transformation. In the limit $L \rightarrow \infty$ the θ_n can be expressed by a continuous function $\theta(k_n)$ and the equal time Green function $G^<(x, 0)$ decays like $x^{-(1+\alpha)}$ for $x \gg 1/k_c$ with $\alpha = 2 \sinh^2[\theta(0)]$, in contrast to x^{-1} in the noninteracting case. Therefore α which in Eq. (16) determines the behaviour of $\langle n_{k,+} \rangle$ near the Fermi point k_F is called the *anomalous dimension*. It can also be expressed in terms of the stiffness constant K

$$\alpha = 2 \sinh^2 \theta(0) = (K - 1)^2 / 2K. \quad (20)$$

For small interaction α is proportional to $\tilde{v}(0)^2$. Only for the special case $\tilde{v}(0) = 0$ the anomalous dimension vanishes and $\langle n_{k,+} \rangle$ has a discontinuity at k_F , the hallmark of Fermi liquid theory [28, 29]. In the generic interacting case one has $\alpha > 0$ and Luttinger's power law Eq. (16) holds for $\alpha < 1$ (see Fig. 2).

At finite temperatures $d\langle n_{k,+} \rangle / dk$ diverges like $T^{\alpha-1}$ at $k = k_F$ for $\alpha < 1$.

From the Fourier transform of $ie^{i\mu t} G^<(0, t)$ one obtains the *local* spectral density $\rho^<(\omega)$, where ω is the energy relative to the chemical potential. For $T = 0$ this leads to a power law suppression [34]

$$\rho^<(\omega) \sim \Theta(-\omega) \left(\frac{-\omega}{v_c k_c} \right)^\alpha \quad (21)$$

of the spectral weight near the chemical potential where Θ denotes the unit step function. This result holds for arbitrary noninteger values of $\alpha \geq 0$ [36]. The energy range over which this asymptotic behaviour can be used depends on the functional form of $\tilde{v}(k)$.

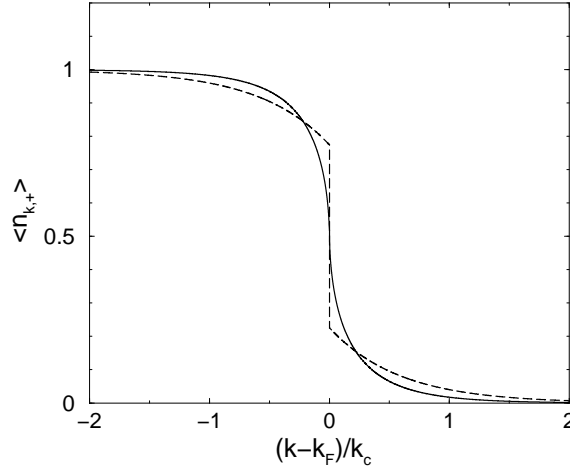


Figure 2. Average occupation $\langle n_{k,+} \rangle$ for different interactions specified by $\sinh \theta(k)$: The full line corresponds to $\sinh^2 \theta(k) = 0.3e^{-2|k|/k_c}$ i.e. the *finite* value $\alpha = 0.6$. The expectation from Fermi liquid theory with a finite jump at k_F is shown as the dashed line for $\sinh^2 \theta(k) = 0.6(|k|/k_c)e^{-2|k|/k_c}$ i.e. *vanishing* anomalous dimension.

For finite temperatures the local spectral weight is nonvanishing also for $\omega > 0$ and $\rho^<(0) \sim T^\alpha$ in the low temperature limit [37].

The spectral function $\rho_+^<(k, \omega)$ relevant for describing angular resolved photoemission is obtained from Eq. (19) by a *double* Fourier transform. Also the total spectral function $\rho_+(k, \omega) \equiv \rho_+^<(k, \omega) + \rho_+^>(k, \omega)$ can be obtained from $\rho_+^<(k, \omega)$ by using the relation $\rho_+^>(k_F + \tilde{k}, \omega) = \rho_+^<(k_F - \tilde{k}, -\omega)$.

For a general k -dependence of $\tilde{v}(k)$ the double transform can be performed analytically only approximately [36]. The exact calculation is possible numerically, e.g. recursively [38, 39, 40]. An exception is provided by $\rho_+^<(k_F, \omega)$. At the Fermi momentum one obtains for $\omega < 0$ asymptotically $\rho_+^<(k_F, \omega) \sim \alpha(-\omega)^{\alpha-1}$, i.e. for $\tilde{v}(0) \neq 0$ a power law divergence as long as $\alpha < 1$. There is *no sharp quasiparticle peak* as in a Fermi liquid. This behaviour as well as the power laws in Eq. (16) and Eq. (21) are the hallmarks of Luttinger liquid behaviour [1].

For $k_F - k_c < k < k_F$ and a *constant* $\tilde{v}(q)$ up to the cutoff k_c the k -resolved spectral function shows a power law singularity at $\omega = v_c(k - k_F)$ if $\alpha < 1/2$ [38]

$$\rho_+^<(k_F + \tilde{k}, \omega) \sim \Theta(-\omega - v_c|\tilde{k}|)(-\omega + v_c\tilde{k})^{\frac{\alpha}{2}-1}(-\omega - v_c\tilde{k})^{\frac{\alpha}{2}}. \quad (22)$$

This result was first obtained assuming $2 \sinh^2 \theta(q) = \alpha e^{-r|q|}$ and $\omega(q) = v_c|q|$ [34]. A comparison of Eqs. (14) and (15) shows that this is not consistent for all q . As the asymptotic analysis in two variables is less developed than for the one-dimensional case it is not even known if for $\tilde{k} < 0$ a power law at $\omega = v_c\tilde{k}$ exists if the derivatives of $\tilde{v}(k)$ are different from zero at $k = 0$ [36]. Another modification of the power law singularities in Eq. (22) for $k \neq k_F$ results from the corrections to the linearization of the dispersion ϵ_k around the Fermi points [41, 42] as further discussed in section 5.

The TL model *including spin* was introduced and solved exactly using fermionic many-body techniques by Dzyaloshinski and Larkin [43]. This included the one-particle Green function $G_{\pm}(x, t)$. The implications for the spectral functions $\rho_{\pm}(k, \omega)$ when spin is included were not discussed.

Using bosonization the step from the spinless to the spinful model is simple. All one has to do is to switch from the boson operators $b_{m,\sigma}^{(\dagger)}$ for the two spin components $\sigma = \uparrow, \downarrow$ to “charge” (c) and “spin” (s) bosons [1, 4]

$$b_{n,c} \equiv \frac{1}{\sqrt{2}} (b_{n,\uparrow} + b_{n,\downarrow}) , \quad b_{n,s} \equiv \frac{1}{\sqrt{2}} (b_{n\uparrow} - b_{n,\downarrow}) . \quad (23)$$

One can write the TL-Hamiltonian $H_{TL}^{(1/2)}$ for spin one-half fermions as [1, 4]

$$H_{TL}^{(1/2)} = H_{TL,c} + H_{TL,s} , \quad (24)$$

where the $H_{TL,a}$ are of the form of Eq. (13) but the interaction matrix elements have the additional label a . The two terms on the rhs of Eq. (24) *commute*, i.e. the charge and spin excitations are completely independent. This is usually called *spin-charge separation* and is another hallmark of LL physics. The “diagonalization” of the two separate parts proceeds exactly as before and the low energy excitations are “massless bosons” $\omega_{n,a} \approx v_a |k_n|$ with the *charge velocity* $v_c = (v_{J_c} v_{N_c})^{1/2}$ and the *spin velocity* $v_s = (v_{J_s} v_{N_s})^{1/2}$. The corresponding two stiffness constants are given by $K_c = (v_{J_c}/v_{N_c})^{1/2}$ and $K_s = (v_{J_s}/v_{N_s})^{1/2}$. The low temperature thermodynamic properties of the TL-model including spin can be expressed in terms of the four velocities $v_{N_c}, v_{J_c}, v_{N_s}, v_{J_s}$ or the four quantities v_c, K_c, v_s, K_s . For spin rotation invariant interactions $K_s = 1$ holds [1, 4].

The one-particle Green functions of the spinful model are given by the square root of the product of the charge and spin part which individually are of the form in Eq. (19) [43, 44, 45]. The anomalous dimension is given by $\alpha = \sinh^2 \theta_c(0) + \sinh^2 \theta_s(0) \equiv \alpha_c + \alpha_s$, where α_s vanishes in the spin rotation invariant case. Again the spectral function $\rho_+(k, \omega)$ can be calculated analytically in the low energy regime for a constant $\tilde{v}(k)$ up to the cutoff k_c . It shows *two* power law singularities [44, 45] for sufficiently small values of the α_a . For $\alpha_s = 0$ the “spin singularity” is determined by the exponent $(2\alpha - 1)/2$ and the “charge singularity” by $(\alpha - 1)/2$. These “peaks” disperse linearly with $k - k_F$. For the modification of the singularities due to the k -dependence of $\tilde{v}(k)$ and the corrections to the linearisation of ϵ_k the same arguments hold as in the spinless case. At the Fermi momentum one again obtains $\rho_+(k_F, \omega) \sim |\omega|^{\alpha-1}$. For the local spectral density Eq. (21) holds also in the spinful model, and the rhs of Eq. (16) also holds for $\langle n_{k\sigma,+} \rangle$.

Not all interesting results for correlation functions of the TL-model can be listed here. The unusual effect of impurities on Luttinger liquids can e.g. be traced back to the $|Q|^{2(K-1)}$ divergence of the static density response function at $k = \pm 2k_F + Q$ for repulsive interactions [34]. This leads to the breakdown of a perturbational analysis for an impurity potential with a weak $\pm 2k_F$ backscattering contribution. The renormalization

group analysis by Kane and Fisher [46] showed that the backscattering potential is a relevant perturbation for repulsive interactions as expected from earlier work by Mattis [47]. The flow to strong coupling implies that the system behaves as if it is split by the impurity into two chains with fixed boundaries at the end. Therefore it is necessary to mention the behaviour of the one-particle Green function close to a boundary. The bosonization for periodic boundary conditions described above has to be modified [48] to describe fixed boundary conditions. For spinless fermions and x close to the boundary $\langle \psi(x, 0) \psi^\dagger(x, t) \rangle$ decays like $(1/t)^{1+\alpha_B}$ in the long time limit, where $\alpha_B = 1/K - 1$ is the boundary exponent. The local spectral function close to the boundary shows a power law $\rho(x, \omega) \sim |\omega|^{\alpha_B}$, where the proportionality factor contains an oscillatory part in the position variable. The fact that in the low temperature limit the linear conductance of a backscattering impurity vanishes like $T^{2\alpha_B}$ [46] can be understood as an end-to-end tunneling between the “split chains”. In contrast to the “bulk” anomalous dimension α the boundary value α_B is proportional to $\tilde{v}(0)$ for small interactions.

A challenging problem is to describe the *disorder* in a Luttinger liquid with a *finite impurity density* [49, 50, 51]. We refer to chapter 9 of Giamarchi’s book for an extended discussion [4].

4. The Luttinger liquid concept

Tomonaga was well aware of the limitations of his approach for more generic two-body interactions (“In the case of force of too short range this method fails” [2]). It was only realized later that the TL model is the fixed point Hamiltonian for a rather general class of models [52, 1, 53, 54]. This emergence of the general Luttinger liquid concept is discussed in this section.

In the opposite limit $k_c \gg k_F$ and a k -independent interaction Tomonaga’s continuum model corresponds to a short range interaction in real space. Then the low energy scattering processes with momentum transfer $\pm 2k_F$ have to be included. They are usually modeled by the *additional* “ g_1 ”-interaction term

$$H_{\text{int}}^{(1)} = \sum_{\sigma, \sigma'} \int (g_{1\parallel} \delta_{\sigma, \sigma'} + g_{1\perp} \delta_{\sigma, -\sigma'}) \psi_{+, \sigma}^\dagger(x) \psi_{-, \sigma'}^\dagger(x) \psi_{+, \sigma'}(x) \psi_{-, \sigma}(x) dx. \quad (25)$$

Introducing a band cutoff Sólyom [52] made a renormalization group (RG) study of this interaction at the one loop level. If the variable s runs from zero to infinity in the process of integrating out degrees of freedom he obtained for spin-independent interactions $g_{i\parallel} = g_{i\perp} = g_i$ ($i = 1, 2$)

$$\frac{dg_1(s)}{ds} = -\frac{1}{\pi v_F} g_1^2(s), \quad \frac{dg_2(s)}{ds} = -\frac{1}{2\pi v_F} g_1^2(s) \quad (26)$$

with the solution $g_1(s) = g_1/[1 + sg_1/(\pi v_F)]$, where g_1 is the starting value. The g_4 -interaction is not renormalized. For $g_1 > 0$ the interactions flow to the *fixed line*

$g_1^* = 0, g_2^* = g_2 - g_1/2$ and the *fixed point Hamiltonian is a TL-model* [52]. This shows the generic importance of the TL-model for repulsive interactions.

For $g_1 < 0$ the flow is to strong coupling. In order to understand the strong coupling regime Luther and Emery bosonized the additional interaction $H_{\text{int}}^{(1)}$ in Eq. (25) and showed that “spin-charge separation” also holds for this model [55]. The charge part stays trivial with massless charge bosons as the elementary interactions. They showed that for a particular value of $g_{1\parallel}$ the exact solution for the spin part of the Hamiltonian is possible using refermionization. The spectrum for the spin excitations is *gapped*. It is generally believed that these properties of *Luther-Emery phases* are not restricted to the solvable parameter values.

Strong coupling phenomena which lead to deviations from LL behaviour can occur in the *lattice models* like the ones discussed in section 2, when for commensurate fillings *Umklapp processes* become important [4]. Here important results for the models in Eqs. (2) and (3) are presented.

For the spinless fermions Eq. (2) the parameters in Eq. (4) for a chain of N sites with periodic boundary conditions and k -values in the first Brillouin zone are given by

$$\epsilon_k = -2t \cos k, \quad v_{k_1, k_2 k_3, k_4} = \frac{2U \cos(k_1 - k_3)}{N} \sum_{m=0, \pm 1} \delta_{k_1+k_2, k_3+k_4+2\pi m} \quad (27)$$

The $m = 0$ term on the rhs of Eq. (27) represents the direct scattering terms and the $m = \pm 1$ terms the Umklapp processes. It is a low energy process in the *half filled* band case $k_F = \pi/2$ discussed here. Renormalization group analysis around the noninteracting fixed point shows that the Umklapp terms are strongly irrelevant which implies that the system is a LL for small $U > 0$ [56]. For $U \gg t > 0$ charge density wave (CDW) order develops in which every other site is occupied in order to avoid the Coulomb penalty. The mapping to the spin model suggests that the transition occurs at $U_c = 2t$ as for $U_c > 2t$ the Ising term dominates. The exact Bethe ansatz solution confirms this and shows that the model at half filling is a LL for $|U| < 2t$. The Luttinger liquid parameters can be obtained using a ground state property and the lowest charge excitation [57]. For repulsive interactions $U > 0$ the value of $K = \pi/[2 \arccos(-U/2t)]$ decreases monotonously from the noninteracting value $K = 1$ to $K = 1/2$ for $U = 2t$, which corresponds to an anomalous dimension $\alpha = 1/4$. In order to reach smaller values than $1/2$ for K the interaction has to have a longer range in real space [4]. The limit $K \rightarrow 0$ is reached by the bare Coulomb interaction as $\tilde{v}(k) \sim \log(1/|k|)$ for $k \rightarrow 0$ and the system is not a LL. The $4k_F$ -harmonic of the density-density correlation function shows a very slow decay almost like in a Wigner crystal [58].

The limit in which the lattice constant and the density go to zero corresponds to the continuum limit. The interaction goes over to a contact interaction. Because of the Pauli principle its effect vanishes and $K \rightarrow 1$. This limit is very different for the Hubbard model Eq. (3) as the onsite interaction is between electrons with *different* spins.

The energy dispersion ϵ_k for the Hubbard model Eq. (3) is the same as in Eq. (27) and the interaction matrix elements $v_{k_1\sigma_1,k_2\sigma_2,k_3\sigma_3,k_4\sigma_4}$ have the same m -sum. The k -independent prefactor is proportional to $\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4}\delta_{\sigma_1,-\sigma_2}$. To show the difference to the spinless model the focus is again on the half filled band case which is metallic for $U = 0$. The limit $U \gg t$ is easy to understand. Each site is singly occupied in order to avoid the Coulomb penalty. In this limit the model can be mapped to the spin model Eq. (1) with $J_x = J_y = J_z = 4t^2/U$ and $h = 0$, i.e. the spin-1/2 Heisenberg antiferromagnet which has gapless excitations [13, 4, 14]. In contrast there is a large gap $\Delta_c \approx U$ for excitations in the charge sector. As the model can be solved exactly by a generalized Bethe ansatz approach this *Mott-Hubbard* gap can be obtained exactly by solving Lieb and Wu's integral equations [23, 14]. It turns out to be finite for *all* $U > 0$. It is exponentially small $\Delta_c \approx (8t/\pi)\sqrt{U/t}\exp(-2\pi t/U)$ for $0 < U \ll t$. This shows that the Umklapp term is no longer irrelevant at the noninteracting fixed point. As the Pauli principle does not influence electrons of opposite spin the RG analysis shows that the Umklapp terms are *marginally relevant* at the noninteracting fixed point [59, 4, 14]. When the band is *not* half filled Umklapp is *not* a low energy process and the Hubbard model is a Luttinger liquid with $K_s = 1$ for all $U > 0$. The LL parameters K_c and v_a ($a = c, s$) can be obtained by numerically solving Lieb and Wu's integral equations. The results show that $K_c \rightarrow 1/2$ for $n \rightarrow 0$ as well as $n \rightarrow 1$ (half filling) for *all* $U > 0$ [60, 59].

For results for various correlation functions we refer to the textbook on the one-dimensional Hubbard model [14].

The discussion of the two lattice models Eqs. (2) and (3) shows the general importance of the Luttinger liquid concept for one-dimensional fermions with repulsive interaction. Only for half filling qualitative deviations occur.

As mentioned in the introduction the focus of this paper is on fermionic systems. *Bosons* in one dimension with repulsive interaction also behave as Luttinger liquids. For them there is an essential difference between the noninteracting and the interacting system [4, 61]. For the nonrelativistic dispersion $\epsilon_k = k^2/(2m)$ the excitation spectrum of the bosonic many body system is linear in $|k|$, i.e. has the typical LL form only if a *finite* interaction is present. In addition to the sound velocity the low energy physics is described by the stiffness constant K which again determines the large distance and long time behaviour of correlation functions. In contrast to fermionic systems the noninteracting bosons correspond to the limit $K \rightarrow \infty$.

Very versatile systems to experimentally test the Luttinger liquid behaviour for bosons in one dimension are ultracold gases in strong optical lattices [5].

Examples for the experimental realization of Luttinger liquid behaviour in quasi-one-dimensional electronic systems are presented in a separate introductory paper

[62]. The comparison of theory and experiment faces the problem that strictly one-dimensional systems are a theoretical idealization. Apart from this even the coupling to an experimental probe presents a nontrivial disturbance of a Luttinger liquid. The coupling between the chains in a very anisotropic 3d compound generally, at low enough temperatures, leads to true *long range order*. The order develops in the phase for which the algebraic decay of the corresponding correlation function of the single chain LL is slowest [59]. This can lead e.g. to charge density wave (CDW), spin density wave (SDW) order or superconductivity. Unfortunately the weak coupling between several LLs or the coupling of a LL to a substrate is theoretically not very well understood [4, 26]. The discussion could easily fill a paper itself.

Carbon nanotubes are one-dimensional metallic systems when the semimetallic carbon sheet is properly wrapped. In contrast to the systems discussed above *two* bands cross the Fermi level like in a two-leg-ladder [4]. Within an approximate treatment of the interaction terms it is possible to describe the system in close analogy to the “regular” Luttinger liquids discussed so far [63, 64].

One-dimensional metals having elementary excitations which propagate along the boundary of a two-dimensional system in one direction only are another type of Luttinger liquids. Wen introduced the concept of *chiral* Luttinger liquids to describe the edge excitations in the fractional quantum Hall states [65, 66].

5. Extensions of the Luttinger liquid concept and outlook

The Luttinger liquid concept has recently been extended further in various ways. An important step was to examine effects beyond Tomonaga’s linearization Eq. (6) by including terms of order $(k - k_F)^2$ (or order $(k - k_F)^3$ for the half filled lattice models) [41, 42]. This makes relaxation processes possible which do not exist in “linear” Luttinger liquids. The interaction modifies e.g. the “narrow box” centered around qv_F in $S(q, \omega)$ discussed in sec. 3 and the k -resolved spectral functions $\rho(k, \omega)$ for $k \neq k_F$. The new “nonlinear Luttinger liquid” phenomenology makes contact to methods developed for describing the x-ray edge singularity of core hole spectra in the presence of a Fermi sea [67, 68]. The many-body dynamics is described using effective models for mobile quantum impurities in a *linear* Luttinger liquid [69, 41, 42].

The temperature dependence of the spectral functions was discussed only briefly so far. In the low temperature regime $k_B T \ll v_c k_c$ the power law behaviour is smoothed out but the anomalous dimension can be recovered using the low energy scaling relation $\rho^<(\omega, k_B T) = T^\alpha F(\omega/k_B T)$ [37, 70, 71]. A new scenario can result in the limit that one of several intrinsic energy scales goes to zero. An example is the half filled Hubbard model discussed in the previous section. In the limit $U \rightarrow \infty$ the exchange coupling $J = 4t^2/U$ goes to zero and with it the spin velocity v_s . In the temperature range

$J \ll k_B T \ll v_c/a_0$ called *spin-incoherent* Luttinger liquid regime the one-particle spectral functions show qualitatively different behaviour from that discussed in section 3 [72, 73]. Details can be found in a review by Fiete [74].

Helical Luttinger liquids are realized in helical conductors e.g. on the edges of topological insulators. Unlike the chiral Luttinger liquid a helical LL does not break time reversal symmetry in these systems with strong spin-orbit interaction. Helical LL exhibit spin-filtered transport where right movers carry spin up and left movers spin down [75, 76]. Yet another type of Luttinger liquid behaviour can be realized in the presence of nuclear moments. A new ordered phase can result by the coupling of these moments to the conduction electrons via the RKKY interaction [77]. The resulting low energy physics was dubbed *spiral* Luttinger liquid by the authors. A detailed comparison of the spectral properties of regular, helical and spiral Luttinger liquids was presented recently [78, 79].

In recent years a very active field of theoretical research is to generalize the description of one-dimensional quantum systems to conditions far from thermal equilibrium. A typical example is a finite (interacting) quantum wire which in the initial state is attached from the left and right to noninteracting leads with differing chemical potentials $\mu_{L(R)}$ and temperatures $T_{L(R)}$ [80, 81]. The theoretical description is usually done using the Keldysh technique [82, 83]. For an even more general class of nonequilibrium states where the initial states of the leads are not of the grand canonical form a new bosonization technique has been developed [84] which uses concepts familiar from “full counting statistics” [85] as well as the x-ray edge problem [67, 68].

Obviously this short introduction cannot cover all the important contributions to the theory of one-dimensional quantum many body systems. As we started the discussion with the spin 1/2 chain we end by mentioning that important insights into the physics of isotropic antiferromagnetic chains of arbitrary spin were obtained using methods of conformal field theory [86, 87, 88].

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